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Elastic-Plastic Orthotropic Multilayered Pipe Deformation Under External Load and Internal Pressure

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Elastic-plastic deformation of a thin-walled pipe, composed of layers with material orthotropic in the elastic and plastic domains, is analyzed. The first material direction for each layer is inclined for an angle ($+\alpha$ and $-\alpha$ successively) with respect to the pipe axis. The yield condition of the material represents a generalization of Hill's criterion to include material hardening. The pipe is loaded by internal pressure and other external loads, and it is supposed that the pipe cross section is free to expand or contract. The derived incremental relations for stress integration, which take into account the stress-strain conditions in the pipe wall, are based on the governing parameter method, developed by the first author, where the problem of implicit integration of inelastic constitutive relations within a time (load) step is reduced to solution of one governing nonlinear equation. Also, the expressions for the tangent elastic-plastic constitutive matrix are derived. One solved simple numerical example demonstrates the main characteristics of the developed algorithm, especially suited for a general elastic-plastic analysis of composite pipes (within the displacement-based finite element method).

Nomenclature

B, N = material constants
C = tangent constitutive matrix
e, e^{in} = total and inelastic strains
p = governing parameter
S = deviatoric stress
$\underline{\beta}$ = internal variable
$\underline{\Delta e}^p$ = increment of plastic strain
$\underline{\sigma}$ = stress

Subscripts

a, b, c = axial, normal, and hoop axis

Superscripts

t = start of time step, left-hand side
$t + \Delta t$ = end of time step, left-hand side

I. Introduction

IN a general finite element analysis calculation of the structural response, with inelastic material behavior, represents a complex problem. Within this task, integration of constitutive relations in an incremental form is one of the most important steps. Also, to achieve a high-convergence rate in equilibrium iterations on the structural level, it is necessary to determine the true tangent moduli.

Significant efforts in last decade have been directed to formulate efficient, reliable, and general computational procedures for integration of inelastic constitutive relations. The implicit stress integration algorithms have been favored over other approaches.[1–9] These algorithms have been implemented to simple and complex material models.

One of the concepts that provides the aforementioned desirable characteristics of the computational procedure is the governing parameter method (GPM), presented in Ref. 1; in Sec. II we outline this concept. In Sec. III we briefly describe the orthotropic elastic-plastic model as a special case of a general anisotropic metal plasticity material,[10] which also represents a generalization of Hill's model.[11] The computational procedure for stress integration within the time (load) step, which takes into account the physical conditions in the thin pipe wall, is presented in Sec. IV. Based on the derived relations for stress calculation and according to GPM, the expressions for the tangent elastic-plastic moduli are derived in Sec. V. In Sec. VI we give a simple numerical example to illustrate the main features of the developed algorithm. Some concluding remarks which point out the applicability of the presented computational procedure to a general analysis of elastic-plastic pipe deformation are given in Sec. VII.

II. Governing Parameter Method for Implicit Stress Integration

The governing parameter method for implicit stress integration, developed in Ref. 1, is applicable to material models where all unknowns at the end of the time step can be expressed in terms of one (governing) parameter, such as in the case of metal plasticity and/or creep[5,10] or for some materials in soil plasticity.[7–9]

Let the known variables at start of time (load) step be

$$ {}^{t}\underline{\sigma}, \quad {}^{t}e, \quad {}^{t}e^{\text{in}}, \quad {}^{t}\underline{\beta}, \quad {}^{t+\Delta t}e \tag{1}$$

whereas the unknowns are

$$ {}^{t+\Delta t}\underline{\sigma}, \quad {}^{t+\Delta t}e^{\text{in}}, \quad {}^{t+\Delta t}\underline{\beta} \tag{2}$$

An example of internal variable is the back stress in Refs. 5 and 6. The implicit stress integration procedure according to the GPM consists in the following steps.

Step 1) Express the unknowns in terms of a governing parameter p,

$$\begin{gathered} {}^{t+\Delta t}\underline{\sigma}({}^{t}\underline{\sigma}, {}^{t}e, {}^{t}e^{\text{in}}, {}^{t}\underline{\beta}, {}^{t+\Delta t}e, p) \\ {}^{t+\Delta t}e^{\text{in}}({}^{t}\underline{\sigma}, {}^{t}e, {}^{t}e^{\text{in}}, {}^{t}\underline{\beta}, {}^{t+\Delta t}e, p) \\ {}^{t+\Delta t}\underline{\beta}({}^{t}\sigma, {}^{t}e, {}^{t}e^{\text{in}}, {}^{t}\underline{\beta}, {}^{t+\Delta t}e, p) \end{gathered} \tag{3}$$

Step 2) Form a governing nonlinear equation

$$f(p) = 0 \tag{4}$$

and solve for ${}^{t+\Delta t}p$.

Step 3) Substitute the solution ${}^{t+\Delta t}p$ into Eq. (3).

According to the above procedure of stress integration we can determine the tangent constitutive matrix C, as follows:

$$C = \frac{\partial\, {}^{t+\Delta t}\underline{\sigma}}{\partial\, {}^{t+\Delta t}e} = \frac{\partial\, {}^{t+\Delta t}\underline{\sigma}}{\partial\, {}^{t+\Delta t}p} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}e} \tag{5}$$

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Derivatives of the governing parameter p with respect to strains ${}^{t+\Delta t}e$ can be obtained by differentiation of the governing equation (4) with respect to ${}^{t+\Delta t}e$; hence,

$$\frac{\partial^{t+\Delta t}p}{\partial^{t+\Delta t}e} = -\left[\frac{\partial f}{\partial^{t+\Delta t}\underline{\sigma}}\frac{\partial^{t+\Delta t}\underline{\sigma}}{\partial^{t+\Delta t}p} + \frac{\partial f}{\partial^{t+\Delta t}e^{\text{in}}}\frac{\partial^{t+\Delta t}e^{\text{in}}}{\partial^{t+\Delta t}p} + \frac{\partial f}{\partial^{t+\Delta t}\underline{\beta}}\frac{\partial^{t+\Delta t}\underline{\beta}}{\partial^{t+\Delta t}p}\right]^{-1}\frac{\partial f}{\partial^{t+\Delta t}e} \tag{6}$$

We note that the governing equation is usually the yield condition at the end of the time step, but it can be formed by some tensorial multiplications, as in case of the effective stress function for creep.[5,6] The governing parameter can be increment of equivalent plastic strain $\Delta\bar{e}_a^P$, as in this paper, an increment of effective creep strain,[5,6] or it can be increment of the volumetric plastic strain,[8] etc.

III. Orthotropic Elastic-Plastic Material Model

A general anisotropic elastic-plastic metal plasticity material model can be described by the yield function of the form[10]

$$f_y = \tfrac{1}{2}N_{ij}(\hat{S}_i - \hat{S}_j)^2 + B_i\hat{S}_i^2 - \tfrac{1}{3}\hat{\sigma}_y^2 = 0 \tag{7}$$

where $\hat{S}_i$ are deviatoric components of tensor that defines the yield surface size (yield surface radius) and $\hat{\sigma}_y$ is the equivalent uniaxial yield stress.

Here we consider a special case of the yield condition (7), which represents a generalization of the Hill's yield condition[11] and corresponds to the orthotropic metal plasticity. The yield condition can be written in the form

$$f_y = \bar{\sigma}_a - \sigma_y = 0 \tag{8}$$

where

$$\bar{\sigma}_a = \left\{\tfrac{3}{2}\left[N_1(S_{xx} - S_{yy})^2 + N_2(S_{xx} - S_{zz})^2 + N_3(S_{yy} - S_{zz})^2 + 2\left(N_{xy}S_{xy}^2 + N_{yz}S_{yz}^2 + N_{xz}S_{xz}^2\right)\right]\right\}^{\frac{1}{2}} \tag{9}$$

is the equivalent (effective) stress, $S_{xx}, S_{yy}, \ldots, S_{xz}$ are deviatoric stresses and $N_1, \ldots, N_{xz}$ are material constants. The material constants can be related to the uniaxial yield stresses X, Y, and Z, in the three material axes and to the shear yield stresses in the three principal material planes Y_{xy}, Y_{yz}, Y_{xz} as

$$\begin{aligned} N_1 &= \tfrac{2}{3}H\sigma_y^2, & N_2 &= \tfrac{2}{3}G\sigma_y^2, & N_3 &= \tfrac{2}{3}F\sigma_y^2 \\ N_{xy} &= \tfrac{1}{3}L\sigma_y^2, & N_{yz} &= \tfrac{1}{3}M\sigma_y^2, & N_{xz} &= \tfrac{1}{3}N\sigma_y^2 \end{aligned} \tag{10}$$

where[11]

$$\begin{aligned} F &= \frac{1}{2}\left(\frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2}\right) \\ G &= \frac{1}{2}\left(\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2}\right) \\ H &= \frac{1}{2}\left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}\right) \\ L &= \frac{1}{2Y_{xy}^2}, \quad M = \frac{1}{2Y_{yz}^2}, \quad N = \frac{1}{2Y_{xz}^2} \end{aligned} \tag{11}$$

The yield stress σ_y, which represents a generalization of the von Mises yield stress for isotropic metal plasticity, can be defined as

$$\sigma_y = \left\{\tfrac{1}{2}\left[\tfrac{1}{3}(X^2 + Y^2 + Z^2) + Y_{xy}^2 + Y_{xz}^2 + Y_{yz}^2\right]\right\}^{\frac{1}{2}} \tag{12}$$

The increment of the equivalent plastic strain $\mathrm{d}\bar{e}_a^P$, which corresponds to the equivalent stress $\bar{\sigma}_a$, is

$$\mathrm{d}\bar{e}_a^P = \left\{\frac{2}{3}\left[\frac{N_1}{\bar{N}}\left(N_3\mathrm{d}e_{xx}^P - N_2\mathrm{d}e_{yy}^P\right)^2 + \frac{N_2}{\bar{N}} \times \left(N_3\mathrm{d}e_{xx}^P - N_1\mathrm{d}e_{zz}^P\right)^2 + \frac{N_3}{\bar{N}}\left(N_2\mathrm{d}e_{yy}^P - N_1\mathrm{d}e_{zz}^P\right)^2 + 2\frac{\left(\mathrm{d}e_{xy}^P\right)^2}{N_{xy}} + 2\frac{\left(\mathrm{d}e_{yz}^P\right)^2}{N_{yz}} + 2\frac{\left(\mathrm{d}e_{xz}^P\right)^2}{N_{xz}}\right]\right\}^{\frac{1}{2}} \tag{13}$$

where $\mathrm{d}e_{xx}^P, \ldots, \mathrm{d}e_{xz}^P$ are increments of plastic strain components, and

$$\bar{N} = N_1N_2 + N_2N_3 + N_1N_3 \tag{14}$$

For a description of the material model it is necessary to define hardening through the uniaxial yield curve,

$$\sigma_y = \sigma_y\left(\bar{e}_a^P\right) \tag{15}$$

with the plastic modulus

$$E_p = \frac{\mathrm{d}\sigma_y}{\mathrm{d}\bar{e}_a^P} \tag{16}$$

IV. Stress Integration Procedure for Elastic-Plastic Pipe Deformation

Consider a multilayered thin-walled pipe as shown in Fig. 1. The local pipe-skin coordinate system consists of the axial axis a, normal axis b to the pipe wall, and hoop axis c. The material axes of a layer are x, y, and z, differing from a, b, and c due to rotation for angle α around the b axis, as shown in the figure. To provide symmetry of the pipe behavior, we suppose that there are an even number of layers, with angles $+\alpha$ and $-\alpha$ successively (two such layers, with axes x, y, and z and $\bar{x}$, $\bar{y}$, and $\bar{z}$ are shown in the figure).

The physical conditions used in further analysis are 1) the hoop stress σ_{cc} is known as the static equivalent of the internal pressure and 2) stress $\sigma_{bb} = 0, \sigma_{bc} = 0$, and strain $\gamma_{bc} = 0$. Condition 1 means that we consider pipe cross sections free to expand or contract. For cross sections where this condition is not fulfilled, as for positions on a pipe structure with flanges, the analysis to be given is not applicable; then we need to implement the condition that the hoop strain is equal to zero, $e_{cc} = 0$, etc. Using conditions 1 and 2, we can write the constitutive relations at end of time (load) step in the form

$$\begin{aligned} {}^{t+\Delta t}e_{aa} - {}^{t}e_{aa}^P - \Delta e_{aa}^P &= F_{11}^E\,{}^{t+\Delta t}\sigma_{aa} + F_{13}^E\,{}^{t+\Delta t}\sigma_{cc} + F_{14}^E\,{}^{t+\Delta t}\sigma_{ab} + F_{16}^E\,{}^{t+\Delta t}\sigma_{ac} \\ {}^{t+\Delta t}\gamma_{ac} - {}^{t}\gamma_{ac}^P - 2\Delta e_{ac}^P &= F_{16}^E\,{}^{t+\Delta t}\sigma_{aa} + F_{36}^E\,{}^{t+\Delta t}\sigma_{cc} + F_{46}^E\,{}^{t+\Delta t}\sigma_{ab} + F_{66}^E\,{}^{t+\Delta t}\sigma_{ac} \\ {}^{t+\Delta t}\gamma_{ab} - {}^{t}\gamma_{ab}^P - 2\Delta e_{ab}^P &= F_{14}^E\,{}^{t+\Delta t}\sigma_{aa} + F_{34}^E\,{}^{t+\Delta t}\sigma_{cc} + F_{44}^E\,{}^{t+\Delta t}\sigma_{ab} + F_{46}^E\,{}^{t+\Delta t}\sigma_{ac} \end{aligned} \tag{17}$$

where ${}^{t+\Delta t}e_{aa}, \ldots, {}^{t}\gamma_{ac}^P$ are strains and plastic strains and F_{ij}^E are components of the elastic compliance matrix corresponding to axes

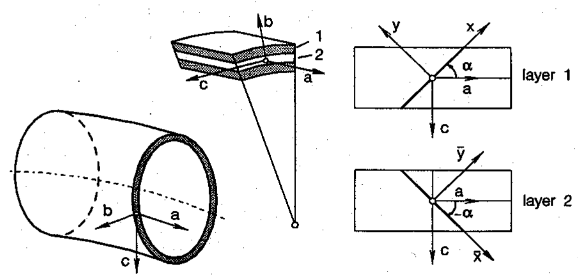

Fig. 1 Multilayered pipe made of orthotropic material.

a, b, and c. In case of a cross section with a flange, ${}^{t+\Delta t}\sigma_{cc}$ is unknown, but the additional condition ${}^{t+\Delta t}e_{cc} = 0$ provides the fourth equation; this condition will not be further analyzed. Using transformation of the compliance matrix $\bar{F}^E_{ij}$ from the material to the pipe axes, according to tensorial transformation of stresses and strains,[12] we obtain

$$\begin{aligned}
F^E_{11} &= c^4\bar{F}_{11} + s^4\bar{F}^E_{22} + s^2c^2\left(2\bar{F}^E_{12} + \bar{F}^E_{44}\right)\\
F^E_{13} &= (c^4+s^4)\bar{F}^E_{12} + s^2c^2\left(\bar{F}^E_{11} + \bar{F}^E_{22} - \bar{F}^E_{44}\right)\\
F^E_{14} &= F^E_{34} = F^E_{46} = 0\\
F^E_{16} &= 2sc\left[-c^2\bar{F}^E_{11} + s^2\bar{F}^E_{22} + (c^2-s^2)\left(\bar{F}^E_{12} + \tfrac{1}{2}\bar{F}^E_{44}\right)\right]\\
F^E_{44} &= s^2\bar{F}^E_{55} + c^2\bar{F}^E_{66}\\
F^E_{36} &= 2sc\left[c^2\bar{F}^E_{22} - s^2\bar{F}^E_{11} - (c^2-s^2)\left(\bar{F}^E_{12} + \tfrac{1}{2}\bar{F}^E_{44}\right)\right]\\
F^E_{66} &= 4s^2c^2\left(\bar{F}^E_{11} + \bar{F}^E_{22} - 2\bar{F}^E_{12}\right) + (c^2-s^2)^2\bar{F}^E_{44}
\end{aligned} \tag{18}$$

where $s = \sin\alpha$ and $c = \cos\alpha$. With these relations we rewrite Eq. (17) as

$$\begin{aligned}
F^E_{11}\,{}^{t+\Delta t}\sigma_{aa} + F^E_{16}\,{}^{t+\Delta t}\sigma_{ac} &= {}^{t+\Delta t}e''_{aa} - \Delta e^P_{aa}\\
F^E_{16}\,{}^{t+\Delta t}\sigma_{aa} + F^E_{66}\,{}^{t+\Delta t}\sigma_{ac} &= {}^{t+\Delta t}e''_{ac} - 2\Delta e^P_{ac}\\
F^E_{44}\,{}^{t+\Delta t}\sigma_{ab} &= {}^{t+\Delta t}e''_{ab} - 2\Delta e^P_{ab}
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
{}^{t+\Delta t}e''_{aa} &= {}^{t+\Delta t}e_{aa} - {}^{t}e^P_{aa} - F^E_{13}\,{}^{t+\Delta t}\sigma_{cc}\\
{}^{t+\Delta t}e''_{ac} &= {}^{t+\Delta t}\gamma_{ac} - {}^{t}\gamma^P_{ac} - F^E_{36}\,{}^{t+\Delta t}\sigma_{cc}\\
{}^{t+\Delta t}e''_{ab} &= {}^{t+\Delta t}\gamma_{ab} - {}^{t}\gamma^P_{ab}
\end{aligned} \tag{20}$$

are known strains.

Further we use the flow rule to express increments of plastic strains $\Delta\bar{e}^P_{ij}$ in the material axes in terms of deviatoric stresses ${}^{t+\Delta t}S_{xx}, \ldots, {}^{t+\Delta t}S_{xz}$. Also, these deviatoric stresses can be expressed in terms of stresses appearing in Eqs. (19) and (20). By transformation of $\Delta\bar{e}^P_{ij}$ to the axes a, b, and c we obtain

$$\begin{aligned}
\Delta e^P_{aa} &= \Delta\lambda\left(\bar{N}_{11}\,{}^{t+\Delta t}\sigma_{aa} + \bar{N}_{16}\,{}^{t+\Delta t}\sigma_{ac} + \bar{N}_{13}\,{}^{t+\Delta t}\sigma_{cc}\right)\\
\Delta e^P_{ac} &= \Delta\lambda\left(\bar{N}_{61}\,{}^{t+\Delta t}\sigma_{aa} + \bar{N}_{66}\,{}^{t+\Delta t}\sigma_{ac} + \bar{N}_{63}\,{}^{t+\Delta t}\sigma_{cc}\right)\\
\Delta e^P_{ab} &= \Delta\lambda\left(s^2N_{yz} + c^2N_{xz}\right){}^{t+\Delta t}\sigma_{ab}
\end{aligned} \tag{21}$$

where $\Delta\lambda$ is a positive scalar. The coefficients $\bar{N}_{ij}$ are

$$\begin{aligned}
\bar{N}_{11} &= \left(c^2 - \tfrac{1}{3}\right)\bar{N}_{xx} + \left(s^2 - \tfrac{1}{3}\right)\bar{N}_{xy} + 2s^2c^2N_{xy}\\
\bar{N}_{16} &= 2sc\left[-\bar{N}_{xx} + \bar{N}_{xy} + (c^2-s^2)N_{xy}\right]\\
\bar{N}_{13} &= \left(s^2 - \tfrac{1}{3}\right)\bar{N}_{xx} + \left(c^2 - \tfrac{1}{3}\right)\bar{N}_{xy} - 2s^2c^2N_{xy}\\
\bar{N}_{61} &= sc\left[\left(c^2 - \tfrac{1}{3}\right)\bar{N}_{yx} + \left(s^2 - \tfrac{1}{3}\right)\bar{N}_{yy} + (c^2-s^2)N_{xy}\right]\\
\bar{N}_{66} &= 2s^2c^2(-\bar{N}_{yx} + \bar{N}_{yy}) + (c^2-s^2)^2N_{xy}\\
\bar{N}_{63} &= sc\left[\left(s^2 - \tfrac{1}{3}\right)\bar{N}_{yx} + \left(c^2 - \tfrac{1}{3}\right)\bar{N}_{yy} - (c^2-s^2)N_{xy}\right]
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
\bar{N}_{xx} &= c^2(N_1 + 2N_2) - s^2(N_1 - N_3)\\
\bar{N}_{xy} &= -c^2(N_1 - N_2) + s^2(N_1 + 2N_3)\\
\bar{N}_{yx} &= -2(N_1 + N_2) + N_3\\
\bar{N}_{yy} &= 2(N_1 + N_3) - N_2
\end{aligned} \tag{23}$$

Finally, substituting Eq. (21) into Eq. (19) we solve for the unknown stresses,

$$\begin{aligned}
{}^{t+\Delta t}\sigma_{aa} &= (1/D_\lambda)\Big[\left({}^{t+\Delta t}e''_{aa} - \Delta\lambda\bar{N}_{13}\,{}^{t+\Delta t}\sigma_{cc}\right)\left(F^E_{66} + 2\Delta\lambda\bar{N}_{66}\right) - \left({}^{t+\Delta t}e''_{ac} - 2\Delta\lambda\bar{N}_{63}\,{}^{t+\Delta t}\sigma_{cc}\right)\left(F^E_{16} + \Delta\lambda\bar{N}_{16}\right)\Big]\\
{}^{t+\Delta t}\sigma_{ac} &= (1/D_\lambda)\Big[\left({}^{t+\Delta t}e''_{ac} - 2\Delta\lambda\bar{N}_{63}\,{}^{t+\Delta t}\sigma_{cc}\right)\left(F^E_{11} + \Delta\lambda\bar{N}_{11}\right) - \left({}^{t+\Delta t}e''_{aa} - \Delta\lambda\bar{N}_{13}\,{}^{t+\Delta t}\sigma_{cc}\right)\left(F^E_{16} + 2\Delta\lambda\bar{N}_{61}\right)\Big]\\
{}^{t+\Delta t}\sigma_{ab} &= \frac{{}^{t+\Delta t}e''_{ab}}{F^E_{44} + 2\Delta\lambda\left(s^2N_{yz} + c^2N_{xz}\right)}
\end{aligned} \tag{24}$$

where

$$D_\lambda = \left(F^E_{11} + \Delta\lambda\bar{N}_{11}\right)\left(F^E_{66} + 2\Delta\lambda\bar{N}_{66}\right) - \left(F^E_{16} + 2\Delta\lambda\bar{N}_{61}\right)\left(F^E_{16} + \Delta\lambda\bar{N}_{16}\right) \tag{25}$$

The stresses in Eq. (24) are expressed in terms of the unknown scalar $\Delta\lambda$. According to Eqs. (9) and (13) we can obtain $\Delta\lambda$ in the form

$$\Delta\lambda = 3/2\left(\Delta\bar{e}^P_a / {}^{t+\Delta t}\bar{\sigma}_a\right) \tag{26}$$

The equivalent stress ${}^{t+\Delta t}\bar{\sigma}_a$ is related to increment $\Delta\bar{e}^P_a$ through the yield curve, therefore, stresses (24) are functions of one parameter $\Delta\bar{e}^P_a$, which, in accordance with the GPM, represents the governing parameter.

Following the computational procedure of Sec. II, we have now the following steps:

1) Suppose $\Delta\bar{e}^P_a$.
2) Determine ${}^{t+\Delta t}\sigma_y$ from the yield curve.
3) Calculate $\Delta\lambda$ from Eq. (26) using ${}^{t+\Delta t}\bar{\sigma}_a = {}^{t+\Delta t}\sigma_y$.
4) Determine ${}^{t+\Delta t}\sigma_{aa}$, ${}^{t+\Delta t}\sigma_{ac}$, and ${}^{t+\Delta t}\sigma_{ab}$ from Eq. (24).
5) Calculate ${}^{t+\Delta t}\bar{\sigma}_a$ according to Eq. (9) by using relations (5) between ${}^{t+\Delta t}S_{ij}$ and stresses (24).
6) Repeat the steps until

$$\left|f\left(\Delta\bar{e}^P_a\right)\right| = \left|{}^{t+\Delta t}\bar{\sigma}_a - {}^{t+\Delta t}\sigma_y\right| < \varepsilon$$

7) Calculate increments of plastic strains according to Eq. (21).

In the calculations we have supposed that the material coefficients $N_1, N_2, \ldots, N_{xz}$ in Eq. (9) are constant, although their change with $\bar{e}^P_a$ can be taken into account by employing data from simple experiments suggested through expressions (10–15).

V. Elastic-Plastic Matrix

For ease of notation we introduce here a one-index notation so that the elastic plastic matrix ${}^{t+\Delta t}C^{EP}_{ij}$ has the form

$${}^{t+\Delta t}C^{EP}_{ij} = \frac{\partial\,{}^{t+\Delta t}\sigma_i}{\partial\,{}^{t+\Delta t}e_j} \qquad i, j = 1, 2, 3 \tag{27}$$

where $i, j = 1, 2$, and 3 represent indices aa, ab, and ac. Following the concept described in Sec. II, we express ${}^{t+\Delta t}C^{EP}_{ij}$ as

$${}^{t+\Delta t}C^{EP}_{ij} = A_{ij} + B_{ii}\Delta\bar{e}^P_{a,j} \qquad \text{no sum on } i \tag{28}$$

where $\Delta\bar{e}^P_{a,j} = \partial(\Delta\bar{e}^P_a)/\partial\,{}^{t+\Delta t}e_j$. The coefficients A_{ij} and B_{ii} are

$$A_{11} = \left(F^E_{66} + 2\Delta\lambda\bar{N}_{66}\right)/D_\lambda$$

$$A_{13} = -\left(F^E_{16} + \Delta\lambda\bar{N}_{16}\right)/D_\lambda$$

$$A_{22} = 1/\left[F^E_{44} + 2\Delta\lambda\left(s^2N_{yz} + c^2N_{xz}\right)\right]$$

$$A_{31} = -\left(F^E_{16} + 2\Delta\lambda\bar{N}_{61}\right)/D_\lambda$$

$$A_{33} = \left(F^E_{11} + \Delta\lambda\bar{N}_{11}\right)/D_\lambda$$

$$
\begin{aligned}
B_{11} = \frac{a_\sigma}{D_\lambda}\big[&-\bar{N}_{13}{}^{t+\Delta t}\sigma_{cc}\big(F^E_{66} + 2\Delta\lambda\bar{N}_{66}\big) + 2\bar{N}_{66}\big({}^{t+\Delta t}e''_{aa} \\
&- \Delta\lambda\bar{N}_{13}{}^{t+\Delta t}\sigma_{cc}\big) + 2\bar{N}_{63}{}^{t+\Delta t}\sigma_{cc}\big(F^E_{16} + \Delta\lambda\bar{N}_{16}\big) \\
&- \bar{N}_{16}\big({}^{t+\Delta t}e''_{ac} - 2\Delta\lambda\bar{N}_{63}{}^{t+\Delta t}\sigma_{cc}\big) - {}^{t+\Delta t}\sigma_{aa}D_{\lambda,\lambda}\big] \\
B_{22} = -2a_\sigma&\big(s^2N_{yz} + c^2N_{xz}\big){}^{t+\Delta t}\sigma_{ab}\big/\big[F^E_{44} + 2\Delta\lambda \\
&\times\big(s^2N_{yz} + c^2N_{xz}\big)\big] \\
B_{33} = \frac{a_\sigma}{D_\lambda}\big[&-2\bar{N}_{63}{}^{t+\Delta t}\sigma_{cc}\big(F^E_{11} + \Delta\lambda\bar{N}_{11}\big) \\
&+ \bar{N}_{11}\big({}^{t+\Delta t}e''_{ac} - 2\Delta\lambda\bar{N}_{63}{}^{t+\Delta t}\sigma_{cc}\big) \\
&+ \bar{N}_{13}{}^{t+\Delta t}\sigma_{cc}\big(F^E_{16} + 2\Delta\lambda\bar{N}_{61}\big) \\
&- 2\bar{N}_{61}\big({}^{t+\Delta t}e''_{aa} - \Delta\lambda\bar{N}_{13}{}^{t+\Delta t}\sigma_{cc}\big) - {}^{t+\Delta t}\sigma_{ac}D_{\lambda,\lambda}\big]
\end{aligned}
$$

and other A_{ij} are equal to zero. The coefficients a_σ and $D_{\lambda,\lambda}$ are

$$
a_\sigma = \frac{1}{{}^{t+\Delta t}\sigma_y}\left(\frac{3}{2} - \Delta\lambda^{t+\Delta t}E_p\right) \tag{29}
$$

and $D_{\lambda,\lambda} = \partial D_\lambda/\partial(\Delta\lambda)$ which follows from Eq. (25).

Next, we differentiate the governing relation

$$
{}^{t+\Delta t}f\big(\Delta\bar{e}^P_a\big) = {}^{t+\Delta t}\bar{\sigma}_a - {}^{t+\Delta t}\sigma_y = 0 \tag{30}
$$

with respect to ${}^{t+\Delta t}e_j$ and, according to Eq. (6), obtain

$$
\big(\Delta\bar{e}^P_{a,j}\big) = -W_j/W_0 \tag{31}
$$

where

$$
\begin{aligned}
W_j &= N'_{xx}b_{1j} + N'_{yy}b_{2j} + N'_{xy}b_{3j} + (-sN'_{yz} + cN'_{xz})A_{2j} \\
W_0 &= N'_{xx}d_{11} + N'_{yy}d_{22} + N'_{xy}d_{33} + (-sN'_{yz} + cN'_{xz})B_{22} \\
&\quad - \tfrac{2}{3}{}^{t+\Delta t}\bar{\sigma}_a{}^{t+\Delta t}E_P
\end{aligned} \tag{32}
$$

The coefficients $N'_{xx}, \ldots, N'_{xz}$ are

$$
\begin{aligned}
N'_{xx} &= N_1\big({}^{t+\Delta t}S_{xx} - {}^{t+\Delta t}S_{yy}\big) + 2N_2\big({}^{t+\Delta t}S_{xx} - {}^{t+\Delta t}S_{zz}\big) \\
&\quad + N_3\big({}^{t+\Delta t}S_{yy} - {}^{t+\Delta t}S_{zz}\big) \\
N'_{yy} &= -N_1\big({}^{t+\Delta t}S_{xx} - {}^{t+\Delta t}S_{yy}\big) + N_2\big({}^{t+\Delta t}S_{xx} - {}^{t+\Delta t}S_{zz}\big) \\
&\quad + 2N_3\big({}^{t+\Delta t}S_{yy} - {}^{t+\Delta t}S_{zz}\big) \\
N'_{xy} &= 2N_{xy}{}^{t+\Delta t}S_{xy}, \qquad N'_{yz} = 2N_{yz}{}^{t+\Delta t}S_{yz} \\
N'_{xz} &= 2N_{xz}{}^{t+\Delta t}S_{xz}
\end{aligned} \tag{33}
$$

whereas b_{ij} and d_{ii} are

$$
\begin{aligned}
b_{1j} &= \big(c^2 - \tfrac{1}{3}\big)A_{1j} - 2scA_{3j} \\
b_{2j} &= \big(s^2 - \tfrac{1}{3}\big)A_{1j} + 2scA_{3j} \\
b_{3j} &= -scA_{3j} - (c^2 - s^2)A_{3j} \\
d_{11} &= \big(c^2 - \tfrac{1}{3}\big)B_{11} - 2scB_{33} \\
d_{22} &= \big(s^2 - \tfrac{1}{3}\big)B_{11} + 2scB_{33} \\
d_{33} &= -scB_{11} - (c^2 - s^2)B_{33}
\end{aligned} \tag{34}
$$

The calculated ${}^{t+\Delta t}C^{EP}$ is symmetric, although that is not obvious from the expressions. In the solved example we give some numerical values of ${}^{t+\Delta t}C^{EP}_{ij}$.

Finally, we note that in the practical finite element application it is necessary to transform the matrix ${}^{t+\Delta t}C^{EP}$ from the local coordinate system of the pipe wall to the local pipe coordinate system, r, s, t indicated in Fig. 1. This is a standard fourth-order tensorial transformation which reduces to a double multiplication of ${}^{t+\Delta t}C^{EP}$ with the matrices employed for stress or strain transformation.[12]

VI. Numerical Example

We consider a simple example: a cantilever pipe loaded by constant internal pressure, bending moment M and transversal force F_T, as shown in Fig. 2a. Geometrical, loading, and material data are given in the same figure. Two layers with angles $\alpha = 30$ and $\alpha = -30$ deg of the first material axis and with unit thickness are taken. We employ the usual beam assumption for distributions of bending strains e_{rr} and γ_{rt} as shown in the figure.

Incremental equilibrium equations for a cross section can be written in the form

$$
\begin{aligned}
{}^{t+\Delta t}K^{(i-1)}_{11}\Delta\varphi'^{(i)}_s + {}^{t+\Delta t}K^{(i-1)}_{13}\Delta\gamma^{(i)}_{rt} &= {}^{t+\Delta t}M - {}^{t+\Delta t}M^{(i-1)}_\sigma \\
{}^{t+\Delta t}K^{(i-1)}_{31}\Delta\varphi'^{(i)}_s + {}^{t+\Delta t}K^{(i-1)}_{33}\Delta\gamma^{(i)}_{rt} &= {}^{t+\Delta t}F_T - {}^{t+\Delta t}F^{(i-1)}_{T\sigma}
\end{aligned} \tag{35}
$$

where i denotes the iteration number, $\Delta\varphi'^{(i)}_s$ and $\Delta\gamma^{(i)}_{rt}$ are increments of the pipe rotation per unit length and of shear strain, ${}^{t+\Delta t}M^{(i-1)}_\sigma$ and ${}^{t+\Delta t}F^{(i-1)}_{T\sigma}$ are bending moment and transversal force corresponding to stresses ${}^{t+\Delta t}\sigma^{(i-1)}$, and ${}^{t+\Delta t}K^{(i-1)}_{11}, \ldots, {}^{t+\Delta t}K^{(i-1)}_{33}$ are stiffnesses that are related to the constitutive matrix ${}^{t+\Delta t}C^{EP(i-1)}$ according to the beam assumptions. In this example we have employed 20 points along the circumference to numerically

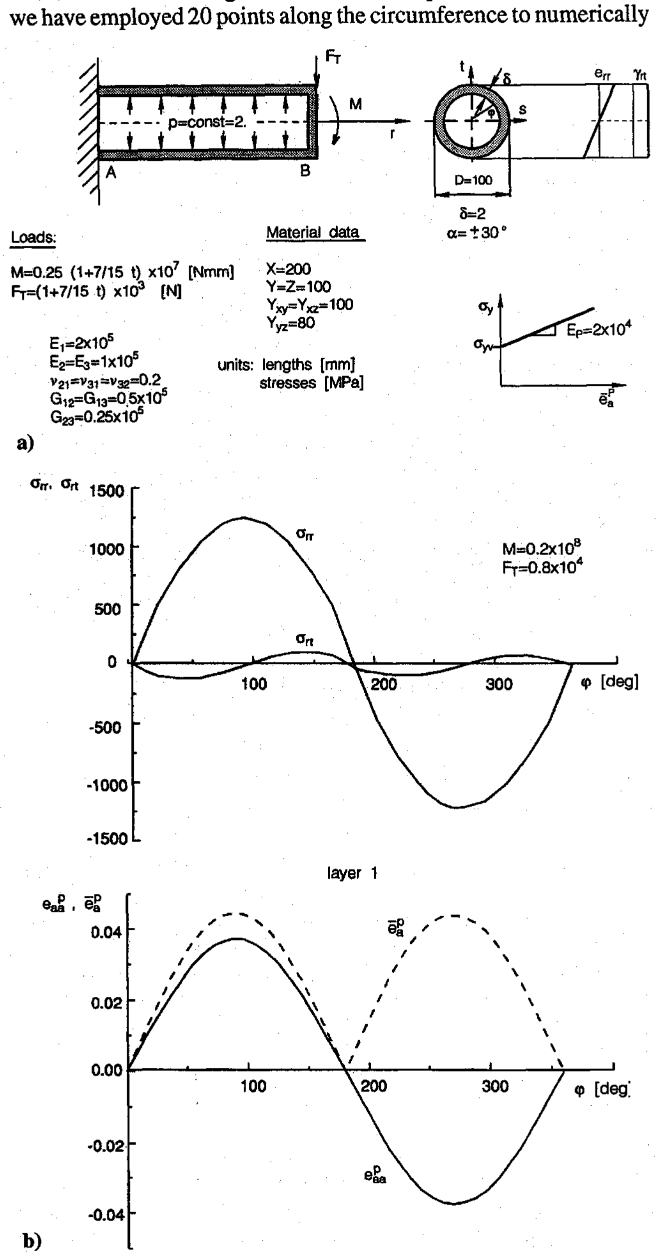

Fig. 2 Multilayered pipe loaded by internal pressure, bending moment, and transversal force: a) geometrical, material, and loading data and b) distribution along circumference, axial stress σ_{rr}, shear stress σ_{rt}, equivalent plastic strain $\bar{e}^P_a$, and axial plastic strain e^P_{aa}.

Table 1 Stresses and strains for the point defined by angle $\varphi = 72$ deg, layer 1, $M = 0.20 \times 10^8$, $F_T = -0.80 \times 10^4$

Strains				
aa, bb, cc	0.42909×10^{-1}	-0.32796×10^{-1}	-0.80577×10^{-2}	
ab, ac	-0.88784×10^{-3}	-0.28848×10^{-3}		
xx, yy, zz	0.30292×10^{-1}	0.45590×10^{-2}	-0.32796×10^{-1}	
xy, yz, xz	-0.43994×10^{-1}	0.62990×10^{-3}	-0.66152×10^{-3}	
Stresses				
aa, cc, ab, ac	0.11972×10^4	0.50000×10^2	-0.75745×10^1	-0.20226×10^3
xx, yy	0.10856×10^4	0.16163×10^3		
xy, yz, xz	-0.39562×10^3	0.37872×10^1	-0.65597×10^1	
Plastic strains				
aa, bb, cc	0.35793×10^{-1}	-0.30301×10^{-1}	-0.54916×10^{-2}	
ab, ac	-0.69848×10^{-3}	0.65649×10^{-3}		
xx, yy, zz	0.25188×10^{-1}	0.51138×10^{-2}	-0.30301×10^{-1}	
xy, yz, xz	-0.36082×10^{-1}	0.47841×10^{-3}	-0.53032×10^{-3}	
Equiv. stress, equiv. plastic strain	0.10028×10^4	0.42524×10^{-1}		

Table 2 Unbalanced energies $\Delta E_{(i)}$ for step 15

Iteration number	1	2	3	4	5
ΔE_i	0.5505×10^2	0.9563×10^1	0.4154×10^{-2}	0.1093×10^{-4}	0.4829×10^{-10}

calculate coefficients in Eq. (35) and we have neglected changes of stresses and strains in the direction of the layer thicknesses.

Starting with $t = 0$ and $\Delta t = 1$, 15 equal steps are used. In Table 1 we give some results for the last step, corresponding to point defined by angle $\varphi = 72$ deg and to layer 1. Distributions along the circumference of stresses σ_{aa} and σ_{rt} and plastic strains e^P_{aa} and $\bar{e}^P_a$ are shown in Fig. 2b. We note that there is symmetry (or antisymmetry) of results with respect to the plane $s = 0$ (for example, symmetry for values σ_{rr}, σ_{rt}, σ_{xx}, e^P_{aa}, e^P_{bb}, e^P_{ab}, and $\bar{e}^P_a$ and antisymmetry in components σ_{ac}, σ_{yz}, e^P_{ac}, and e^P_{yz}). Results close to these shown in the figure are obtained by modeling the pipe by composite shell finite elements of the finite element (FE) package PAK.[14] Maximum strains are around 4%, which can be considered as the upper limit for assumption that changes of the cross-sectional size and shape are neglected.

In Table 2 we give the unbalanced energy during iterations, calculated as

$$\Delta E_{(i)} = \left(M - M_\sigma^{(i-1)}\right)\Delta\varphi_s'^{(i)} + \left(F_T - F_{T\sigma}^{(i-1)}\right)\Delta\gamma_{rt}^{(i)} \qquad (36)$$

From the table we see the quadratic convergence rate, which follows from the tangent character of elastic-plastic matrix derived in Sec. V. The same rate of convergence is obtained for other load steps.

VII. Concluding Remarks

The developed algorithm for stress integration is applicable to a general elastic-plastic analysis of multilayered thin-walled pipe structures, for cross sections free to expand or contract. The condition that the pipe wall is thin is important, since in the proposed computational procedure changes of strains through the wall thickness are neglected. It is an application of the governing parameter method according to which the stress calculation reduces to solution of one nonlinear equation. The local conditions at the pipe wall, relying on the assumptions that there is zero-normal stress through the pipe skin and that the hoop stress is equal to the internal pressure-static equivalent, are exactly satisfied.

The computational procedure is numerically efficient and accurate, since it is implicit, and applicable to large strain increments in a load step and to orthotropic elastic-plastic material with hardening characteristics. The derived expressions for the tangent moduli, obtained by the appropriate differentiation of the governing relations, provide high convergence rate in equilibrium iterations on the structural level.

The presented computational algorithm is especially suitable for application in a general-purpose finite element programs for nonlinear analysis, such as Ref. 14, where the pipe element is considered as a special case of composite beam with arbitrary cross section.[15]

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